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ON THE RHEOLOGICAL INSTABILITY OF AN ELASTIC DAMAGING MEDIUM*

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A medium is examined that contains damage of the microcrack type scattered over the volume, whose number and dimensions can vary under the action of applied stresses. Such materials include brittle rocks, certain metal alloys, glass, etc. To describe the behaviour of such media a model of continuum fracture of elastic bodies /1/ is used based on the local balance between the effective surface energy of the microcracks and the cumulative elastic energy of the material surrounding the microcracks. The constraints on the allowable strain values imposed by the Hadamard condition /2/, which is a necessary condition for the correctness of any dynamic or quasistatic problems, are investigated in an isothermal approximation. These constraints play the part of a strength criterion that is closely associated with the internal structure of the rheological relationships used.

It is shown that violation of the Hadamard condition, identifiable with the rheological instability of the material, is accompanied by the formation of stationary surfaces of strain discontinuity, where, unlike an elastic-plastic dilating material /3, 4/, the origination of rheological instability is possible for the model being used in both the loading process and in unloading of the material. The orientation of the

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strain discontinuity surface is found with respect to the principal axes of the strain tensor.

Following /3/ and considering stationary strain discontinuity surfaces as boundaries of narrow strain localization zones that are typical structural elements of natural bodies and loaded laboratory specimens /5, 6/, the relationships obtained can be interpreted as generation conditions and spatial characteristics of macro-discontinuities in an elastic damaging medium.

Two fundamental approaches can be extracted in the mathematical modelling of the behaviour of a medium for which processes of generation and evolution of numerous defects scattered over the volume are characteristic. The first relies on methods of the theory of plasticity of dilatant materials. The characteristic features of the strain mechanism are: microcrack opening and closing, their generation and growth accompanied by friction slippage of the edges, taken into account by using the dependence of the yield point on the hydrostatic pressure, the non-associated law of plastic flow, taking account of inelastic volume compressibility, etc. The governing equations of such media were developed in /3, 7-9/. The other approach is associated with the theory of continual scattered fracture of solids /10-12/, also called the theory of damagability, continuum theory of defects, etc. We shall later use the model of scattered fracture /1/ that is a continuum analogue of the Griffith approach in the mechanics of an isolated macrocrack /13/ to describe small isothermal strains of a homogeneous and initially-isotropic elastic damaging medium.

1. Fundamental equations of a damaging medium. Let ε be the small strain tensor of a stress-free reference configuration $\varkappa$ in a configuration $\chi(t)$ of a body at a time t, $\varepsilon' = \varepsilon - {}^1/_3\mathbf{I}(\mathbf{I}:\varepsilon)$ is the deviator of the tensor ε, $\mathbf{I}_1 = \mathbf{I}:\varepsilon$, $J = (\varepsilon':\varepsilon')^{1/2}$ where $\mathbf{I}$ is the unit tensor of second rank and the colon denotes a double scalar product. Let $\mathbf{N} = \varepsilon'/J$ be a normalized strain tensor deviator for which the relationships

$$\mathbf{N}:\mathbf{I} = 0, \quad \mathbf{N}:\mathbf{N} = 1 \tag{1.1}$$

hold.

We will characterize damagability of a material by the scalar quantity $\omega \ll 1$ that is, the ratio between the mass densities in the reference body configuration $\varkappa$ and the unloaded body configuration $\varkappa^*$. As in pasticity theory, the unloaded configuration $\varkappa^*$ obtained from $\chi(t)$ by removal of the stresses in each body element, is not identical with $\varkappa$, if inelastic strains preceded the unloading.

Let us select the potential of the medium in the simplest form

$$\rho u(\varepsilon, \omega) = {}^1/_2 K I_1^2 + \mu J^2 - \alpha_p \omega I_1 - \alpha_s J\omega \tag{1.2}$$
$$(\rho, K, \mu, \alpha_p, \alpha_s = \text{const} > 0)$$

where ρ is the mass density, K and μ are the bulk compression and shear moduli, α_p, α_s are parameters characterizing the decrease in the elastic energy density as the damagability ω increases because of partial unloading of the material in the neighbourhood of microcracks as their number or size increases.

An expression for the stress tensor /1/ follows from (1.2)

$$\mathrm{T} = \rho\frac{\partial u(\varepsilon, \omega)}{\partial \varepsilon} = (KI_1 - \alpha_p\omega)\mathbf{I} + \left(2\mu - \frac{\alpha_s\omega}{J}\right)\varepsilon' \tag{1.3}$$

We will also specify the effective surface energy density $u_f(\omega)$ by the simplest expression

$$\rho u_f(\omega) = \rho u_f^0 + \gamma\omega + {}^1/_2\beta\omega^2 \tag{1.4}$$
$$(u_f^0, \gamma, \beta = \text{const} > 0)$$

The resistance to the growth of fracturing in this case equals

$$\rho G \equiv \rho\partial u_f(\omega)/\partial\omega = \gamma + \beta\omega \tag{1.5}$$

Hence, the phenomenological meaning of the constants γ and β is recognized.

Two processes are possible in the material under consideration: passive, in which $\omega^\cdot = 0$ and active, in which $\omega^\cdot \neq 0$. Moreover, assuming the microcracks not to be closed up in the time intervals under consideration, and energy to be expended in defect formation, we will assume that $\omega \geqslant 0$, $\omega^\cdot > 0$ in the active process. Then when there are no distributed heat sources and any kinds of energy, the condition of local balance of elastic and surface energies in the active process can be written in the form

$$\partial u(\varepsilon, \omega)/\partial\omega + G(\omega) = 0, \quad \omega \geqslant 0, \quad \omega^{\cdot} > 0$$

The relation $\omega(\varepsilon)$ in the active process follows from (1.2) and (1.5)

$$\omega = (\alpha_p I_1 + \alpha_s J - \gamma)/\beta, \quad \omega \geqslant 0, \quad \omega^{\cdot} > 0 \tag{1.6}$$

If $\alpha_p I_1 + \alpha_s J - \gamma < 0$ or the rate of change of the strain is such that $\alpha_p I_1^{\cdot} + \alpha_s J^{\cdot} < 0$, then the passive process $(\omega^{\cdot} = 0)$ is realized, which is either a process of deformation of the undamaged material $(\omega \equiv 0)$ or an unloading process in which $\omega = \omega_*$, where ω_* is the value of the damagability at the time of passage from active loading to the unloading process under consideration.

It is seen from (1.3) that when there is no damagability $(\omega \equiv 0)$ the connection between the stress and strain agrees with Hooke's law. During unloading $(\omega = \omega_* = \mathrm{const})$ the stresses are connected with the strains by the relation

$$\mathbf{T} = K I_1 \mathbf{I} + 2\mu\varepsilon' - \mathbf{T}_*, \quad \mathbf{T}_* \equiv \alpha_p \omega_* \mathbf{I} + \alpha_s \omega_* \mathbf{N} \tag{1.7}$$

Denoting the particle mass velocity vector by v and the gradient over the space variable $\mathbf{x} \in \chi(t)$, by ∇ we can write the system of differential equations of the dynamics of an elastic damaging medium in the variables $(\mathbf{v}, \varepsilon)$ in the form

$$\rho\mathbf{v}^{\cdot} - \nabla\cdot\mathbf{T}(\varepsilon, \omega(\varepsilon)) = 0, \quad 2\varepsilon^{\cdot} - \nabla\otimes\mathbf{v} - (\nabla\otimes\mathbf{v})^T = 0 \tag{1.8}$$

It is here taken into account that ω is either a constant, or is connected with the strain ε by the relationships (1.6). We neglect mass forces.

Using (1.3), the variation of the stress tensor in the active process can be represented in the form

$$\delta\mathbf{T} = \rho\mathbf{L}(\varepsilon, \omega) : \delta\varepsilon$$

where

$$\mathbf{L} = \frac{\partial^2 u(\varepsilon, \omega(\varepsilon))}{\partial\varepsilon\otimes\partial\varepsilon} + \frac{\partial^2 u}{\partial\omega\partial\varepsilon} \otimes \frac{\partial\omega}{\partial\varepsilon} \tag{1.9}$$

is a tensor of the fourth rank for the hypoelastic coefficients that are symmetric in the two first and last subscripts, i.e., $L_{ijab} = L_{jiab} = L_{ijba}$.

Therefore, the system of dynamic equations in the active process can be written in the form of a quasilinear system

$$\mathbf{v}^{\cdot} - \mathbf{L}(\varepsilon) \vdots (\nabla\otimes\varepsilon) = 0, \quad 2\varepsilon^{\cdot} - \nabla\otimes\mathbf{v} - (\nabla\otimes\mathbf{v})^T = 0 \tag{1.10}$$

where the symbol $(\vdots)$ denotes a triple scalar product so that the equality

$$(\mathbf{a}_0\otimes\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3) \vdots (\mathbf{b}_1\otimes\mathbf{b}_2\otimes\mathbf{b}_3) = \mathbf{a}_0 \prod_{i=1}^{3} (\mathbf{a}_i\cdot\mathbf{b}_i)$$

holds for arbitrary tetrads and triads.

In the passive process when the particle damagability $\omega = \omega_*$ is invariant in time $(\omega_*^{\cdot} = 0)$, but variable in space, the system of differential equations is identical with the equations of the linear theory of elasticity of an inhomogeneous material

$$\mathbf{v}^{\cdot} - \mathbf{L}_*(\varepsilon) \vdots (\nabla\otimes\varepsilon) = -\frac{\partial^2 u(\varepsilon, \omega_*)}{\partial\omega\,\partial\varepsilon}\cdot\nabla\omega_*, \quad 2\varepsilon^{\cdot} - \nabla\otimes\mathbf{v} - (\nabla\otimes\mathbf{v})^T = 0 \tag{1.11}$$

$$(\mathbf{L}_*(\varepsilon) = \partial^2 u(\varepsilon, \omega_*)/\partial\varepsilon\otimes\partial\varepsilon)$$

2. *Propagation of weak discontinuities.* Let $\psi(\mathbf{x}, t) = 0$ be the equation of a singular weak discontinuity surface and $c = -\partial\psi/\partial t/|\nabla\psi|$, $\mathbf{v} = \nabla\psi/|\nabla\psi|$ the velocity of propagation and the normal to this surface /2, 14/. Let $\mathbf{V}$ and $\mathbf{E}$ denote the amplitude of jumps normal to the surface $\psi(\mathbf{x}, t) = 0$ of derivatives of the velocity $\mathbf{v}(\mathbf{x}, t)$ and the strain tensor $\varepsilon(\mathbf{x}, t)$. We will use the geometric and kinematic conditions on the surface of the weak discontinuity /2/, that follow from the condition of continuity of $\mathbf{v}, \varepsilon$ on the surface under consideration

$$[\mathbf{v}^{\cdot}] = -c\mathbf{V}, \quad [\nabla\otimes\mathbf{v}] = \mathbf{v}\otimes\mathbf{V}$$

$$[\varepsilon^{\cdot}] = -c\mathbf{E}, \quad [\nabla\otimes\varepsilon] = \mathbf{v}\otimes\mathbf{E}$$

Let the material on both sides of this surface be in an active or passive loading state and, therefore, the components of the tensor $\mathbf{L}(\varepsilon)$ are continuous together with $c\,\mathbf{v}, \varepsilon$. It then follows from system (1.10)

$$c\mathbf{V} + \mathbf{L} : (\mathbf{v} \otimes \mathbf{E}) = 0, \quad 2c\mathbf{E} + \mathbf{v} \otimes \mathbf{V} + \mathbf{V} \otimes \mathbf{v} = 0 \tag{2.1}$$

For a given surface $\psi(\mathbf{x}, t) = 0$ and a given tensor $\mathbf{L}(\varepsilon)$ Eqs.(2.1) are a system of homogeneous linear equations in the quantities $\mathbf{V}, \mathbf{E}$. A non-trivial solution of this system exists in addition to the solution $\mathbf{V} = 0, \mathbf{E} = 0$ if and only if $\psi = 0$ is a characteristic surface /14/ whose propagation velocity c is such that the determinant of the matrix of coefficients of system (2.1) vanishes for any given direction $\mathbf{v}$ of surface propagation. In this case the solution of system (1.10) can be continued through the surface $\psi = 0$ in a non-unique manner, with and without a discontinuity in the normal derivatives.

It can be seen from (2.1) that any stationary surface $\psi(\mathbf{x}) = 0$ with velocity $c \equiv 0$ is a characteristic surface, where the quantity $\mathbf{V}$ thereon equals zero.

To find the non-stationary characteristic surface $(c \not\equiv 0)$, we multiply the first equation in (2.1) by $2c$ and the second, on the left in a tensor manner by $\mathbf{v}$ and we use the operator $\mathbf{L}$:. After subtracting the second equation from the first we arrive at a homogenous linear equation

$$(c^2\mathbf{I} - \mathbf{v}\cdot\mathbf{L}\cdot\mathbf{v})\cdot\mathbf{V} = 0 \tag{2.2}$$

when account is taken of the above-mentioned symmetry of $\mathbf{L}$.

Multiplying (2.2) scalarly on the left by $\mathbf{V}$ we find

$$(\mathbf{v} \otimes \mathbf{V}) : \mathbf{L} : (\mathbf{V} \otimes \mathbf{v}) = c^2(\mathbf{V}\cdot\mathbf{V}) > 0 \tag{2.3}$$

Therefore, the necessary condition for hyperbolicity of system (1.10), the reality of the propagation velocities of characteristic surfaces, reduces to a condition for the tensor of fourth rank $\mathbf{L}$ is positive-definite in dyads of a particular kind, one of whose components is the arbitrary non-zero vector $\mathbf{v}$, while the second is the right eigenvector $\mathbf{V}$ of the acoustic tensor $\mathbf{A}(\varepsilon, \mathbf{v}) \equiv \mathbf{v}\cdot\mathbf{L}(\varepsilon)\cdot\mathbf{v}$, corresponding to the positive eigennumber $c^2 > 0$. Since $\mathbf{V} = \mathbf{V}(\mathbf{v})$, condition (2.3) is generally a weaker condition

$$(\mathbf{v} \otimes \mathbf{b}) : \mathbf{L} : (\mathbf{b} \otimes \mathbf{v}) > 0, \quad \forall \mathbf{b} \neq 0, \ \mathbf{a} \neq 0 \tag{2.4}$$

that expresses the property that $\mathbf{L}$ is positive-definite on arbitrary dyads and which is often called the Hadamard condition, the condition of strong ellipticity /15/, and *SE* is the condition /2/.

The inequalities (2.3) and (2.4) are equivalent for the special case of a symmetric acoustic tensor $\mathbf{A} = \mathbf{A}^T$, that holds for the model under consideration.

Indeed, it follows from the symmetry of $\mathbf{A}$ that among the right eigenvectors that are also in this case left eigenvectors, an orthogonal basis $\mathbf{V}_i, i = 1, 2, 3$, can be selected. Any vector $\mathbf{b}$ can then be represented in the form

$$\mathbf{b} = \sum_{i=1}^{3} b_i\mathbf{V}_i$$

Taking account of the orthogonality of $\mathbf{V}_i$ we have

$$(\mathbf{v} \otimes \mathbf{b}) : \mathbf{L} : (\mathbf{b} \otimes \mathbf{v}) = \Big(\sum_{i=1}^{3} b_i\mathbf{V}_i\Big) \cdot \mathbf{A} \cdot \Big(\sum_{j=1}^{3} b_j\mathbf{V}_j\Big) = \sum_{i=1}^{3} b_i^2\mathbf{V}_i \cdot \mathbf{A} \cdot \mathbf{V}_i$$

Hence, the equivalence of conditions (2.3) and (2.4) follows.

We will regard the state of the material particle governed by the strain tensor as ε^0, rheologically unstable if a vector $\mathbf{v}_0(\varepsilon^0)$, exists such that the velocity of propagation of a non-stationary weak discontinuity wave vanishes in the direction $\mathbf{v}_0$ i.e., $c(\varepsilon^0, \mathbf{v}_0(\varepsilon^0)) = 0$.

Hence, it follows directly that $\det \mathbf{A}(\varepsilon^0, \mathbf{v}_0(\varepsilon^0)) = 0$; a non-trivial solution $\mathbf{V}_0(\varepsilon^0, \mathbf{v}_0)$ exists for the homogeneous equation $\mathbf{A}(\varepsilon^0, \mathbf{v}_0(\varepsilon^0)) \cdot \mathbf{V}_0 = 0$; the quadratic form (2.3) is degenerate in the dyad $\mathbf{v}_0 \otimes \mathbf{V}_0$, i.e.,

$$(\mathbf{v}_0 \otimes \mathbf{V}_0) : \mathbf{L}(\varepsilon^0) : (\mathbf{V}_0 \otimes \mathbf{v}_0) = 0$$

Turning to the second of Eqs.(2.1) it can be seen that $\mathbf{V} \to \mathbf{V}_0 \neq 0$, as $c \to 0$, while the amplitude $\mathbf{E}$ of the jump in the normal derivative of the strain tensor increases without limit, i.e., the weak discontinuity becomes strong on which the velocity is continuous while the strain undergoes a discontinuity.

3. ***Conditions and modes of appearance of rheological instability.*** We will examine the conditions for the origination and orientation of strain discontinuity surfaces. In the active process $(\omega \geqslant 0, \ \omega^{\cdot} > 0)$ an expression for the tensor of the hypoelastic coefficients

$$\rho \mathbf{L}(\boldsymbol{\varepsilon}, \omega(\boldsymbol{\varepsilon})) = (\Lambda - M)\mathbf{I}\otimes\mathbf{I} + 2M\mathbf{1} - \xi(\mathbf{I}\otimes\mathbf{N} + \mathbf{N}\otimes\mathbf{I}) - \eta\mathbf{N}\otimes\mathbf{N} \tag{3.1}$$
$$\varphi \equiv \alpha_s\omega/J,\ M \equiv \mu - {}^1/_2\varphi,\ \Lambda \equiv \lambda + \mu - \alpha_p{}^2/\beta - {}^1/_6\varphi,\ \xi \equiv \alpha_p\alpha_s/\beta,\ \eta \equiv \alpha_s{}^2/\beta - \varphi$$
$$K \equiv \lambda + {}^2/_3\mu,\ \mathbf{1} \equiv {}^1/_2(\delta_a{}^i\delta_b{}^j + \delta_b{}^i\delta_a{}^j)\,\mathbf{э}_i\otimes\mathbf{э}_j\otimes\mathbf{э}^a\otimes\mathbf{э}^b$$

follows from relationships (1.2) and (1.6).

Here and henceforth, $\mathbf{1}$ is a unit tensor of the fourth rank, $\delta_a{}^i$ is the Kronecker delta, $\mathbf{э}_i, \mathbf{э}^a$ are vectors of the natural and dual bases. The equalities

$$\mathbf{a}\cdot\mathbf{1}\cdot\mathbf{b} = {}^1/_2(\mathbf{b}\otimes\mathbf{a} + (\mathbf{a}\cdot\mathbf{b})\mathbf{I}),\quad \mathbf{1}:\mathbf{B} = {}^1/_2(\mathbf{B} + \mathbf{B}^T)$$

hold for any vectors $\mathbf{a}$ and $\mathbf{b}$ and the tensor of second rank $\mathbf{B}$.

The acoustic tensor corresponding to (3.1) is written in the form

$$\rho\mathbf{A}(\boldsymbol{\varepsilon}, \omega(\boldsymbol{\varepsilon}), \mathbf{v}) = M\mathbf{I} + \Lambda\mathbf{v}\otimes\mathbf{v} - \xi(\mathbf{v}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{v}) - \eta\mathbf{n}\otimes\mathbf{n},\quad \mathbf{n} \equiv \mathbf{N}\cdot\mathbf{v}$$

while (2.2) for non-stationary weak discontinuity waves will be

$$\{S\mathbf{I} - \Lambda\mathbf{v}\otimes\mathbf{v} + \xi(\mathbf{v}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{v}) + \eta\mathbf{n}\otimes\mathbf{n}\}\cdot\mathbf{V} = 0,\quad S \equiv \rho c^2 - M \tag{3.2}$$

Multiplying (3.2) scalarly by the vectors $\mathbf{v}, \mathbf{n}, \mathbf{v}\times\mathbf{n}$ where the symbol $\times$ denotes the vector product, we obtain a system of three equations

$$\begin{gathered}(S - \Lambda + \xi(\mathbf{v}\cdot\mathbf{n}))\,x + (\xi + \eta(\mathbf{v}\cdot\mathbf{n}))\,y = 0\\ -(\Lambda(\mathbf{v}\cdot\mathbf{n}) - \xi(\mathbf{n}\cdot\mathbf{n}))\,x + (S + \xi(\mathbf{v}\cdot\mathbf{n}) + \eta(\mathbf{n}\cdot\mathbf{n}))\,y = 0\\ Sz = 0\end{gathered} \tag{3.3}$$

where $x \equiv \mathbf{v}\cdot\mathbf{V},\ y \equiv \mathbf{n}\cdot\mathbf{V},\ z \equiv (\mathbf{v}\times\mathbf{n})\cdot\mathbf{V}$.

If $\mathbf{v}$ is not an eigenvector of the normalized deviator $\mathbf{N}$ such that $\mathbf{v}\times\mathbf{n} \neq 0$, system (3.3) is equivalent to (3.2). A non-trivial solution of system (3.3) is possible under the condition

$$S\ (S^2 - 2pS - q) = 0 \tag{3.4}$$

where

$$\begin{gathered}2p(\boldsymbol{\varepsilon}, \mathbf{v}) = \lambda + \mu - \beta^{-1}\mathbf{v}\cdot(\alpha_p\mathbf{I} + \alpha_s\mathbf{N})^2\cdot\mathbf{v} + \varphi(\mathbf{v}\cdot\mathbf{N}^2\cdot\mathbf{v} - {}^1/_6)\\ q(\boldsymbol{\varepsilon}, \mathbf{v}) = \zeta(\mathbf{v}\times\mathbf{n})^2 = \zeta(\mathbf{v}\cdot\mathbf{N}^2\cdot\mathbf{v} - (\mathbf{v}\cdot\mathbf{N}\cdot\mathbf{v})^2),\ \zeta \equiv \xi^2 + \eta\Lambda\end{gathered} \tag{3.5}$$

The roots of (3.4) are

$$\rho c_{1,2}{}^2 = M + p \pm \sqrt{p^2 + q},\qquad \rho c_3{}^2 = M \tag{3.6}$$

Degeneration in c_3 sets in for strains governed by the condition

$$\varphi\ (I_1, J) \equiv \alpha_s\omega\ (I_1, J)/J = 2\mu \tag{3.7}$$

In this case the vector $\mathbf{v}$ is arbitrary while $\mathbf{V}$ is orthogonal to a plane drawn through the vectors $\mathbf{v}$ and $\mathbf{n}$ since $x = y = 0$ follows from the first two equations in (3.2).

We will now find the vector that makes the quantity $\rho c_2{}^2(\boldsymbol{\varepsilon}, \mathbf{v})$ reach an extremum. Without loss of generality $\mathbf{v}$ can be assumed to be a unit vector; consequently, we will seek the extremum by the method of Lagrange multipliers. We write the Lagrange function in the form

$$\Phi(\boldsymbol{\varepsilon}, \mathbf{v}) = \rho c_2{}^2(\boldsymbol{\varepsilon}, \mathbf{v}) - {}^1/_2\varkappa\mathbf{v}\cdot\mathbf{v}$$

where $\varkappa$ is the unknown Lagrange multiplier.

Taking (3.6) into account, the extremum condition is written in the form

$$\frac{\partial\Phi}{\partial\mathbf{v}} = \frac{-1}{2\sqrt{p^2+q}}\left(2S_2\frac{\partial p}{\partial\mathbf{v}} + \frac{\partial q}{\partial\mathbf{v}}\right) - \varkappa\mathbf{v},\quad S_2 \equiv \rho c_2{}^2 - \mu$$

Multiplying this equation scalarly by $\mathbf{v}$, we obtain

$$\varkappa = \frac{-1}{2\sqrt{p^2+q}}\left(2S_2\frac{\partial p}{\partial\mathbf{v}} + \frac{\partial q}{\partial\mathbf{v}}\right)\cdot\mathbf{v}$$

Taking this expression and relationships (3.5) into account, the extremum conditions are written in the form

$$(\mathbf{I}-\mathbf{v}\otimes\mathbf{v})\cdot\mathbf{B}\cdot\mathbf{v}=0 \tag{3.8}$$
$$\mathbf{B}\equiv S_2\{\varphi\mathbf{N}^2-\beta^{-1}(\alpha_p\mathbf{I}+\alpha_s\mathbf{N})^2\}+\zeta\{\mathbf{N}^2-2(\mathbf{v}\cdot\mathbf{N}\cdot\mathbf{v})\,\mathbf{N}\}$$

It follows from relationships (3.8) that $\mathbf{v}$ is an eigenvector of the symmetric tensor $\mathbf{B}$. By virtue of the polynomial representation of $\mathbf{B}$ as a function of $\mathbf{N}$, the eigenvectors of the tensor $\mathbf{N}$ are eigenvectors of $\mathbf{B}$. The converse assertion is generally not true since eigenvectors for $\mathbf{B}$ but not for $\mathbf{N}$ as well, can exist.

If a coordinate system is used whose orthonormalized basis coincides with the principal axes of the tensor $\mathbf{N}$ at the point under consideration, then (3.8) is written in this basis in the form of three scalar equations

$$\{(B_1-B_2)\,\nu_2^2+(B_1-B_3)\,\nu_3^2\}\,\nu_1=0 \quad (1\ 2\ 3) \tag{3.9}$$

where B_1, B_2, B_3 are the eigenvalues of $\mathbf{B}$.

It follows from (3.9) that the non-trivial solution $\mathbf{v}\cdot\mathbf{v}=1$ exists in three cases: 1) $\mathbf{v}$ is an eigenvector of the tensor $\mathbf{N}$, the eigenvalues of the tensor $\mathbf{B}$ are distinct; 2) the vector $\mathbf{v}$ belongs to a plane passing through two eigenvectors of the tensor $\mathbf{N}$, two eigenvalues of the tensor $\mathbf{B}$ are identical; 3) the tensor $\mathbf{B}$ is global.

Let us examine each case in greater detail. Let $\mathbf{e}_i$ $(i=1,2,3)$ denote the orthonormalized basis in agreement with the principal axes of the stress deviator. Then $\mathbf{N}=N_1\mathbf{e}_1\otimes\mathbf{e}_1+N_2\mathbf{e}_2\otimes\mathbf{e}_2+N_3\mathbf{e}_3\otimes\mathbf{e}_3$, where N_i are eigennumbers of the normalized stress deviator. To fix our ideas, let $\mathbf{v}=\mathbf{e}_1$ in the first case. Then $\mathbf{N}\cdot\mathbf{v}=N_1\mathbf{v}$, $\mathbf{v}\cdot\mathbf{N}\cdot\mathbf{v}=N_1$, $\mathbf{v}\cdot\mathbf{N}^2\cdot\mathbf{v}=N_1^2$. Taking into account that $\mathbf{v}\times\mathbf{N}\cdot\mathbf{v}=0$, we find the propagation velocity c_2 of the surface of weak discontinuity: $\rho c_2^2=M$ for $p\geqslant 0$; $\rho c_2^2=M+2p$ for $p<0$. It hence follows that for

$$\lambda+2\mu\,(N_1^2+{}^1/_3)\geqslant\beta^{-1}\,(\alpha_p+\alpha_sN_1)^2$$

rheological instability sets in our $\varphi=2\mu$. The orientation of the surfaces of strain discontinuity can be arbitrary.

For $p<0$, which is equivalent to the condition

$$\lambda+\mu-\beta^{-1}\,(\alpha_p+\alpha_sN_1)^2+\varphi\,(N_1^2-{}^1/_6)<0$$

the minimum perturbation velocity vanishes for strains governed by the equation

$$\lambda+2\mu-\beta^{-1}\,(\alpha_p+\alpha_sN_1)^2+\varphi\,(N_1^2-{}^2/_3)=0 \tag{3.10}$$

The vector $\mathbf{V}$, characterizing the amplitude of the weak discontinuity is identical in this case with the vector $\mathbf{v}$, apart from sign, which enables us to call this mode of appearance of the rheological instability a tension (compression) strain discontinuity.

It follows from (3.10) that the case under consideration can be realized for $N_1^-<N_1<N_1^+$ where $N_1^\pm=(-\alpha_p\pm\sqrt{(\lambda+2\mu)\beta})/\alpha_s$. If $N_1^+<-\sqrt{{}^2/_3}$, i.e., $\lambda+2\mu<\beta^{-1}\,(\alpha_p-\alpha_s\sqrt{{}^2/_3})^2$, the material becomes rheologically unstable at once, as soon as the cumulative damagability process starts.

Turning to the second modification and setting $\nu_1^2+\nu_2^2=1$, $\nu_3=0$, it can be seen that the third equation of system (3.9) is satisfied identically while $B_1=B_2$ and

$$\nu_1^2=\frac{1}{2}\left(1-\zeta^{-1}S_2\,\frac{2\xi-\eta N_3}{N_1-N_2}\right),\quad \nu_2^2=1-\nu_1^2 \tag{3.11}$$

follows from the first two relationships.

The minimum root φ of the equation

$$\rho c_2^2\equiv M+p-\sqrt{p^2+q}=0 \tag{3.12}$$

defines a line in the plane (I_1, J) at whose points the material becomes rheologically unstable. Since $p-\sqrt{p^2+q}\leqslant 0$ for all p and all $q\geqslant 0$, the degeneration sets in for $M>0$.

The amplitude of the discontinuity is characterized by normal and tangential components.

The third case, when the tensor $\mathbf{B}$ is global and all three components of the vector $\mathbf{v}$ are non-zero is realized only for a particular form of the state of strain, uniaxial strain

$$\boldsymbol{\varepsilon}=\varepsilon\mathbf{e}_2\otimes\mathbf{e}_2,\quad N=\pm\sqrt{{}^2/_3}\,\mathbf{e}_2\otimes\mathbf{e}_2\mp(\mathbf{e}_1\otimes\mathbf{e}_1+\mathbf{e}_3\otimes\mathbf{e}_3)/\sqrt{6}$$

In this case the components of the extremal normal vector satisfy the equation

$$2\nu_2^2-(\nu_1^2+\nu_3^2)=(\zeta-\eta S_2\mp 2\sqrt{6}\xi S_2)/(2\zeta) \tag{3.13}$$

which yields the surface of a circular cone with axis $\mathbf{e}_2$. The semi-apex angle Ψ of this cone is given by the expression

$$\sin^2\Psi={}^2/_3-{}^1/_3\zeta^{-1}\,(\eta\pm 2\xi\sqrt{6})S_2$$

which is identical with the first equation of (3.11) for $N_1^2={}^2/_3$, $N_2^2=N_3^2={}^1/_6$.

To prove the relationship (3.13), we note that the global part of the tensor $\mathbf{N}^2$ equals

$^1/_3$I because of the identitites (1.1). This means that the deviator of the tensor B defined by the second relationship of (3.8) equals zero under the condition

$$\{\zeta(\mathbf{v}\cdot\mathbf{N}\cdot\mathbf{v})+2\xi S_2\}\,\mathbf{N}=(\zeta-\eta S_2)(\mathbf{N}^2-{}^1/_3\mathbf{I}) \tag{3.14}$$

Hence it follows, when the operator N: operates on both sides of the equality, that

$$\zeta(\mathbf{v}\cdot\mathbf{N}\cdot\mathbf{v})=(\zeta-\eta S_2)\,\mathbf{N}^3:\mathbf{I}-2\xi S_2$$

Substituting this expression into (3.14), for $\zeta-\eta S_2\neq 0$ we arrive at the equation

$$(\mathbf{N}^3:\mathbf{I})\,\mathbf{N}=\mathbf{N}^2-{}^1/_3\mathbf{I} \tag{3.15}$$

defining the particular kinds of strains for which realization of the case under consideration is possible.

Taking into account that $\mathbf{N}^3:\mathbf{I}=3N_1N_2N_3$, holds in the principal axes of the tensor N, the tensor Eq.(3.15) can be written in these axes in the form of three scalar equations

$$(1-3N_2N_3)\,N_1^2={}^1/_3 \quad (1\ 2\ 3)$$

Taking (1.1) into account, the equation $({}^5/_2-3N_1^2)\,N_1^2={}^1/_3$ with the solutions $N_1^2={}^2/_3$ ahd $N_1^2={}^1/_6$ hence follows.

By virtue of the symmetry of the system of equations, there are exactly the same solutions for N_2 and N_3.

The question arises which of the three cases is ralized. We will assume that the mode of appearance of rheological instability that is realized is the one that corresponds to the least value of the parameter $\varphi\equiv\alpha_s\omega(I_1,J)/J$. Indeed, for $J>0$ lines passing through the point $(\gamma/\alpha_p,0)$ with slope $k_i=\alpha_p\alpha_s/(\varphi_i\beta-\alpha_s^2)$ corresponds to the quantities $\varphi=\varphi_i$, $\varphi_i=$ const $(i=1,2)$ in the (I_1,J) plane. During deformation with any continuous trajectory starting at the point $(0,0)$, the first will intersect a line forming the maximum positive angle with the I_1 axis, i.e., corresponding to the minimum value of φ.

We will now examine the conditions for rheological instability of a material in a passive process. In this case

$$\rho\mathbf{L}(\boldsymbol{\varepsilon}',\omega_*)=(\lambda+{}^1/_3\varphi_*)\,\mathbf{I}\otimes\mathbf{I}+(\mu-{}^1/_2\varphi_*)\,\mathbf{1}+\varphi_*\mathbf{N}\otimes\mathbf{N} \tag{3.16}$$

Eq.(2.2) is written in the form

$$\{S_*\mathbf{I}-\Lambda_*\mathbf{v}\otimes\mathbf{v}-\varphi_*\mathbf{n}\otimes\mathbf{n}\}\cdot\mathbf{V}=0 \tag{3.17}$$

$$M_*\equiv\mu-{}^1/_2\varphi_*,\ \Lambda_*\equiv\lambda+\mu-{}^1/_6\varphi_*,\ \mathbf{n}\equiv\mathbf{N}\cdot\mathbf{v},\ S_*\equiv\rho c^2-M_*$$

The condition for (3.17) to have a non-trivial solution is

$$S_*(S_*^2-2p_*S_*-q_*)=0$$

$$p_*\equiv{}^1/_2(\Lambda_*+\varphi_*\mathbf{n}\cdot\mathbf{n}),\quad q_*\equiv\varphi_*\Lambda_*(\mathbf{v}\times\mathbf{n})^2$$

which results in expressions analogous to (3.6) for the propagation velocities of waves of weak discontinuity in a material under passive strain.

If the expression

$$4(p_*^2+q)=\Lambda_*^2+\varphi_*^2(\mathbf{n}\cdot\mathbf{n})^2+2\varphi_*\Lambda_*(\mathbf{n}\cdot\mathbf{n})\cos 2\alpha$$

is used, where α is the angle between the vectors v and n and the inequality

$$-|\Lambda_*|\leqslant\Lambda_*\cos 2\alpha\leqslant|\Lambda_*|$$

from which there follows

$$^1/_4(|\Lambda_*|-\varphi_*\mathbf{n}\cdot\mathbf{n})\leqslant p_*^2+q_*\leqslant{}^1/_4(|\Lambda_*|+\varphi_*\mathbf{n}\cdot\mathbf{n})$$

then the condition for real wave propagation velocities to exist can be written in the form

$$M_*\equiv\mu-{}^1/_2\varphi_*>0,\quad \Lambda_*+M_*\equiv\lambda+2\mu-{}^2/_3\varphi_*>0 \tag{3.18}$$

For $\mu>0$, $\lambda+2\mu>0$ from which it follows that undamaged material $(\omega\equiv 0)$ is deformed stably, unlike damaged material being unloaded. Indeed, for any damagability level $\omega_*>0$ a shear intensity $J>0$ is found for which one of the relationships (3.18) will cease to hold. If $K>0$ then rheological instability sets in for a shear intensity $J=\alpha_s\omega_*/2\mu$, that makes the stress deviator vanish. This mode of rheological buckling with removal of the shear stresses can occur for arbitrary orientation of the strain discontinuity surfaces.

4. Examples. We will examine the conditions for rheological instability to occur for certain characteristic kinds of state of strain.

Global Strain Tensor. In this case $\boldsymbol{\varepsilon}=I_1\mathbf{I}$, $J=0$, the normalized deviator N is undefined.

Turning directly to (3.1) for the tensor L it can be seen that the coefficients Λ, M, η will be unbounded for $\omega \neq 0$. This means that the undamaged material will lose its continuity as soon as $I_1 = \gamma/\alpha_p$. In other words, under multilateral tension the relationship $I_1 \leqslant \gamma/\alpha_p$ is the traditional criterion for the strength of a material. We note that multilateral compression $(I_1 < 0)$ does not cause cumulative damage of the medium under consideration.

Uniaxial Strain $\varepsilon = \varepsilon e_2 \otimes e_2$. In this case

$$I_1 = \varepsilon, \quad J = |\varepsilon| \sqrt{2/3}, \quad N = \varkappa (\sqrt{2/3}\, e_2 \otimes e_2 - (e_1 \otimes e_1 + e_3 \otimes e_3)/\sqrt{6})$$

where $\varkappa = \operatorname{sign} \varepsilon$. Turning to condition (3.10) for the formation of strain discontinuity planes perpendicular to the principal axes of the tensor N, we find

$$\varphi^{(1)} = 2\{\lambda + 2\mu - \beta^{-1}(\alpha_p - \varkappa\alpha_s/\sqrt{6})^2\}$$

For strain discontinuity surfaces at an angle to the principal axes of the tensor N, the minimum value $\varphi^{(2)}$ is given by (3.12) in which the quantities ν, p, q are determined in conformity with (3.5) and (3.13) by the expressions

$$\nu_1^2 + \nu_3^2 = {}^1/_3(1 - \theta_t), \quad \nu_2^2 = {}^1/_3(2 + \theta_t), \quad q = {}^1/_6(2 + \theta_t)(1 - \theta_t)$$

$$p = \Lambda + 2\varkappa\xi(1 - \theta_t)/\sqrt{6} - {}^1/_2\eta(1 + {}^1/_3\theta_t)$$

$$\theta_t \equiv \zeta^{-1} M(\eta + 2\varkappa\xi\sqrt{6})/2$$

Depending on the specific values of the parameters of the medium, both the first and second cases can be realized. For example, for $\lambda = \alpha_p = \alpha_s = \beta = \mu$ the values are $\varphi^{(1)} = 2.03$, $\varphi^{(2)} = 1.65$, i.e., the rheological instability of the material under uniaxial tension appears in the form of the formation of conical strain discontinuity surfaces. On the other hand, for $\alpha_p = \alpha_s = \beta = \mu$, $\lambda = 0.2\mu$ layers of discontinuity surfaces perpendicular to the boundary are formed first for $\varphi^{(1)} \simeq 0.43$ which can be interpreted as the prototype of separation cracks in compressed rocks /16, 17/.

Pure Shear $\varepsilon = \varepsilon(e_1 \otimes e_1 - e_2 \otimes e_2)$, $\varepsilon > 0$. In this case $I_1 \equiv 0$, $J = \varepsilon\sqrt{2}$, $N_1 = -N_2 = 1/\sqrt{2}$, $N_3 = 0$. The least value of φ corresponding to tension strain discontinuity surfaces equals $\varphi^{(1)} = 6(\lambda + 2\mu - \beta^{-1}(\alpha_p + \alpha_s/\sqrt{2})^2)$ by virtue of (3.10). The appearance of a rheological instability in the mode of strain discontinuity planes close to planes of maximum tangential stress in orientation corresponds to the least root $\varphi^{(2)}$ of (3.12) in which $\nu_{1,2} = {}^1/_2 \pm \theta_s$, $2p = \Lambda - {}^1/_2\eta - 2\sqrt{2}\xi\theta_s$, $q = \zeta({}^1/_2 - 2\theta_s)$, where $\theta_s \equiv \sqrt{2}\xi\zeta^{-1}M$. As for uniaxial strain, both the first and the second cases can be realized depending on the specific values of the parameters of the medium. Thus $\varphi^{(1)} = 0.51$, $\varphi^{(2)} = 1.5$ for $\lambda = \mu = \alpha_p = \alpha_s = \beta$, i.e., strain discontinuity planes perpendicular to the tension axes are formed first. For $\lambda = 1, 2\mu$, $\alpha_p = \alpha_s = \beta = \mu$ $\varphi^{(1)} = 2.01$, $\varphi^{(2)} = 1.85$, holds, i.e., a relatively small change in the moduli can result in a change in the mode of appearance of the rheological instability.

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DUAL FORMULATIONS OF THE BOUNDARY-ELEMENTS METHOD. APPLICATION TO ELASTICITY THEORY PROBLEMS FOR INHOMOGENEOUS BODIES*

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Alternative variational formulations are considered for the boundary-elements method (BEM) that utilize the formulation of minimization problem of boundary functionals and generalized Trefftz functions of linear elasticity theory /1/. The variational solutions are approximated by using boundary potentials with the desired density: the formulation in displacements (line) in place of the interpolation considered earlier of the double layer potential (DLP) density uses interpolation on the boundary element (BE) of the simple layer potential (SLP) density according to the nodal values of the displacements; the dual formulation is interpolation on the BE of the PLP density according to the nodal values of the stresses.

It is best to use the formulation for solving problems of elasticity theory with mixed boundary conditions, contact problems. In particular, the dual formulation turns out to be effective in solving problems for elastic media with discontinuous elasticity coefficients (piecewise-homogeneous); adjoint conditions must be realized in the corresponding variational problem for both the displacement vector and for the stress vector on the surface of discontinuity of the coefficients. The results obtained in /1/ and in this paper are compared with the results arising from other BEM formulations.

1. Duality of the kinematically allowable displacements and statically allowable stresses resulting from the Lagrange-Castigliano principle /2, 3/ is known in linear elasticity theory. A corresponding assertion for surface displacements and stresses follows from dual variational principles for the boundary functionals in problems with bilateral and unilateral constraints on the boundary /4, 5/. The connectedness of the dual formulations of the variational problems (the explicit connection between the variables of the problems in terms of the governing relationships on the boundary) results in identical systems of boundary equations of the Ritz process.

As in /1/ we will give a brief description of the direct BEM formulation on the basis of a problem for a boundary functional

$$\min_{\varphi\in D} E(\varphi), \quad F(\varphi)=\int_S \varphi\mathbf{t}^{(\nu)}(\varphi)\,ds-2\int_S \varphi\mathbf{t}^{(\nu)}(\mathbf{u}^*)\,ds \tag{1.1}$$
$$D(\varphi)=\left\{\varphi \mid \mathbf{A}\varphi(x)=0,\quad x\in G,\quad \int_G \varphi\,dG=\int_G \operatorname{rot}\varphi\,dG=0\right\}$$

Here φ is the displacement vector, $\mathbf{A}$ is a vector operator of isotropic homogeneous elasticity theory, $G\subset E_m$ $(m=2,3)$ is a bounded domain with a sufficiently smooth boundary S with external normal ν, $\mathbf{t}^{(\nu)}(\mathbf{u}^*)$ is the vector of the given stresses at points of S.

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